

SOME NEW INEQUALITIES FOR $(h-s)_{1,2}$ -CONVEX FUNCTIONS VIA FURTHER PROPERTIES

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ABSTRACT. In this paper, we establish some new inequalities of the Hermite-Hadamard like for class of $(h-s)_{1,2}$ -convex functions which are ordinary, super-multiplicative or similarly ordered and nonnegative.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. The following double inequality:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is well-known in the literature as Hadamard's inequality for convex mapping. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f . Both inequalities hold in the reversed direction if f is concave.

Definition 1. [See [2]] We say that $f : I \rightarrow \mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$(1.2) \quad f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

Definition 2. [See [3]] Let $s \in (0, 1]$. A function $f : (0, \infty) \rightarrow [0, \infty]$ is said to be s -convex in the second sense if

$$(1.3) \quad f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in (0, b)$ and $t \in [0, 1]$.

In 1978, Breckner introduced s -convex functions as a generalization of convex functions in [6]. Also, in that one work Breckner proved the important fact that the set valued map is s -convex only if the associated support function is s -convex function in [7]. A number of properties and connections with s -convex in the first sense are discussed in paper [3]. Of course, s -convexity means just convexity when $s = 1$.

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Definition 3. [See [1]] We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a $P-$ function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$, we have

$$(1.4) \quad f(tx + (1 - t)y) \leq f(x) + f(y).$$

Definition 4. [See [4]] Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function. We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is an $h-$ convex function or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $\alpha \in [0, 1]$ we have

$$(1.5) \quad f(\alpha x + (1 - \alpha)y) \leq h(\alpha)f(x) + h(1 - \alpha)f(y).$$

If inequality (1.5) is reversed, then f is said to be $h-$ concave, i.e., $f \in SV(h, I)$. Obviously, if $h(\alpha) = \alpha$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h, I) = Q(I)$; if $h(\alpha) = 1$, $SX(h, I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$.

Furthermore, in [5] Bombardelli and Varošanec wrote some generalizations of the Hermite-Hadamard inequalities and some properties of functions H and F .

More about those inequalities can be found in a number of papers (for example: see [4, 8, 9, 10, 13, 16]).

Definition 5. [See [15]] Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ is an $(h - s)_1$ -convex function in the first sense, or that f belong to the class $SX((h - s)_1, I)$, if f is non-negative and for all $x, y \in [0, \infty) = I$, $s \in (0, 1]$, $t \in [0, 1]$ we have

$$(1.6) \quad f(tx + (1 - t)y) \leq h^s(t)f(x) + (1 - h^s(t))f(y).$$

If inequality (1.6) is reversed, then f is said to be $(h - s)_1$ -concave function in the first sense, i.e., $f \in SV((h - s)_1, I)$.

Definition 6. [See [15]] Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ is an $(h - s)_2$ -convex function in the second sense, or that f belong to the class $SX((h - s)_2, I)$, if f is non-negative and for all $u, v \in [0, \infty) = I$, $s \in (0, 1]$, $t \in [0, 1]$ we have

$$(1.7) \quad f(tu + (1 - t)v) \leq h^s(t)f(u) + h^s(1 - t)f(v).$$

If inequality (1.7) is reversed, then f is said to be $(h - s)_2$ -concave function in the second sense, i.e., $f \in SV((h - s)_2, I)$.

Obviously, in (1.7), if $h(t) = t$, then all s -convex functions in the second sense belongs to $SX((h - s)_2, I)$ and all s -concave functions in the second sense belongs to $SV((h - s)_2, I)$, and it can be easily seen that for $h(t) = t$, $s = 1$, $(h - s)_2$ -convexity reduces to ordinary convexity defined on $[0, \infty)$. Similarly, in (1.6), if $h(t) = t$, then all s -convex functions in the first sense belongs to $SX((h - s)_1, I)$ and all s -concave functions in the first sense belongs to $SV((h - s)_1, I)$, and it can be easily seen that for $h(t) = t$, $s = 1$, $(h - s)_1$ -convexity reduces to ordinary convexity defined on $[0, \infty)$.

Example 1. [See [15]] Let $h(t) = t$ be a function and let the function f be defined as following;

$$f : [2, 4] \rightarrow \mathbb{R}^+, \quad f(x) = \ln x.$$

Then f is non-convex and non- h -convex function, but it is $(h-s)_2$ -convex function.

Definition 7. [See [4]] A function $h : J \rightarrow \mathbb{R}$ is said to be a super-multiplicative function if

$$(1.8) \quad h(xy) \geq h(x)h(y)$$

for all $x, y \in J$.

If inequality (1.8) is reversed, then h is said to be a sub-multiplicative function. If the equality holds in (1.8), then h is said to be a multiplicative function.

Definition 8. [See [11]] A function $h : J \rightarrow \mathbb{R}$ is said to be a super-additive function if

$$(1.9) \quad h(x+y) \geq h(x) + h(y)$$

for all $x, y \in J$.

Definition 9. [See [12]] Two functions $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ are said to be similarly ordered, shortly f s.o. g , if

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for every $x, y \in X$.

Theorem 1. [See [9]] Let $f \in SX(h_1, I)$, $g \in SX(h_2, I)$, $a, b \in I$, $a < b$, be functions such that $fg \in L_1([a, b])$, and $h_1h_2 \in L_1([0, 1])$, then the following inequality holds

$$\frac{1}{b-a} \int_a^b f(x)g(x) dx \leq M(a, b) \int_0^1 h_1(t)h_2(t) dt + N(a, b) \int_0^1 h_1(t)h_2(1-t) dt$$

where

$$\begin{aligned} M(a, b) &= f(a)g(a) + f(b)g(b) \\ N(a, b) &= f(a)g(b) + f(b)g(a). \end{aligned}$$

Motivated by the information given above, main purpose of this paper is to give some inequalities under the special assumptions of h -convex functions by using fairly elementary analysis. We also give some applications to special means. Throughout the paper we will imply $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

2. MAIN RESULTS

We will start with the following result for $(h-s)_1$ -convex functions.

Theorem 2. Let $f, g \in SX((h-s)_1, I)$ and h is a positive function. If $fg \in L_1[a, b]$, then we have the following inequality;

$$\begin{aligned}
(2.1) \quad & \frac{1}{b-a} \int_a^b f^{\frac{x-a}{b-a}}(x) g^{\frac{b-x}{b-a}}(x) dx \\
& \leq \frac{f(b)}{(b-a)^2} \int_a^b (x-a) h^s\left(\frac{x-a}{b-a}\right) dx + \frac{f(a)}{(b-a)^2} \int_a^b (x-a) \left(1 - h^s\left(\frac{x-a}{b-a}\right)\right) dx \\
& \quad + \frac{g(b)}{(b-a)^2} \int_a^b (b-x) h^s\left(\frac{x-a}{b-a}\right) dx + \frac{g(a)}{(b-a)^2} \int_a^b (b-x) \left(1 - h^s\left(\frac{x-a}{b-a}\right)\right) dx
\end{aligned}$$

for all $s \in (0, 1]$.

Proof. Since f, g are $(h-s)_1$ -convex functions on I , we have

$$\begin{aligned}
f(tb + (1-t)a) & \leq h^s(t)f(b) + (1-h^s(t))f(a) \\
g(tb + (1-t)a) & \leq h^s(t)g(b) + (1-h^s(t))g(a).
\end{aligned}$$

For $t \in [0, 1]$, we can write

$$\begin{aligned}
f^t(tb + (1-t)a) & \leq [h^s(t)f(b) + (1-h^s(t))f(a)]^t \\
g^{1-t}(tb + (1-t)a) & \leq [h^s(t)g(b) + (1-h^s(t))g(a)]^{1-t}.
\end{aligned}$$

By multiplying the above inequalities, we get

$$\begin{aligned}
(2.2) \quad & f^t(tb + (1-t)a) g^{1-t}(tb + (1-t)a) \\
& \leq [h^s(t)f(b) + (1-h^s(t))f(a)]^t [h^s(t)g(b) + (1-h^s(t))g(a)]^{1-t}.
\end{aligned}$$

Recall the General Cauchy Inequality (see [14], Theorem 3.1), let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for every positive real numbers x and y , we always have

$$\alpha x + \beta y \geq x^\alpha y^\beta.$$

By applying General Cauchy Inequality to the right hand side of the inequality (2.2), we have

$$\begin{aligned}
& f^t(tb + (1-t)a) g^{1-t}(tb + (1-t)a) \\
& \leq t [h^s(t)f(b) + (1-h^s(t))f(a)] + (1-t) [h^s(t)g(b) + (1-h^s(t))g(a)].
\end{aligned}$$

By integrating the resulting inequality with respect to t , over $[0, 1]$, we obtain

$$\begin{aligned}
& \int_0^1 f^t(tb + (1-t)a) g^{1-t}(tb + (1-t)a) dt \\
& \leq \int_0^1 t [h^s(t)f(b) + (1-h^s(t))f(a)] dt + \int_0^1 (1-t) [h^s(t)g(b) + (1-h^s(t))g(a)] dt.
\end{aligned}$$

Computing the above integrals and by changing of the variable $tb + (1-t)a = x$, $(b-a)dt = dx$, we get the desired result. \square

Corollary 1. (1) In Theorem 2, if we choose $x = a$, we get

$$\frac{1}{b-a} \int_a^b g(x) dx \leq g(b) h^s(0) + g(a) (1 - h^s(0))$$

(2) In Theorem 2, if we choose $x = b$, we get

$$\frac{1}{b-a} \int_a^b f(x) dx \leq f(b) h^s(1) + f(a) (1 - h^s(1))$$

(3) In Theorem 2, if we choose $x = \frac{a+b}{2}$, we get

$$\frac{1}{b-a} \int_a^b \sqrt{f(x)g(x)} dx \leq \frac{f(b) + g(b)}{2} h^s\left(\frac{1}{2}\right) + \frac{f(a) + g(a)}{2} \left(1 - h^s\left(\frac{1}{2}\right)\right).$$

A similar result will obtain in the following Theorem for $(h-s)_2$ -convex functions.

Theorem 3. Let $f, g \in SX((h-s)_2, I)$ and h is a positive function. If $fg \in L_1[a, b]$, then we have the following inequality;

$$\begin{aligned} (2.3) \quad & \frac{1}{b-a} \int_a^b f^{\frac{x-a}{b-a}}(x) g^{\frac{b-x}{b-a}}(x) dx \\ & \leq \frac{f(b)}{(b-a)^2} \int_a^b (x-a) h^s\left(\frac{x-a}{b-a}\right) dx + \frac{f(a)}{(b-a)^2} \int_a^b (x-a) h^s\left(\frac{b-x}{b-a}\right) dx \\ & \quad + \frac{g(b)}{(b-a)^2} \int_a^b (b-x) h^s\left(\frac{x-a}{b-a}\right) dx + \frac{g(a)}{(b-a)^2} \int_a^b (b-x) h^s\left(\frac{b-x}{b-a}\right) dx \end{aligned}$$

for all $s \in (0, 1]$.

Proof. The proof is immediately follows from the proof of the above Theorem, but now f, g are $(h-s)_2$ -convex functions on I . \square

Corollary 2. (1) In Theorem 3, if we choose $x = a$, we get

$$\frac{1}{b-a} \int_a^b g(x) dx \leq g(b) h^s(0) + g(a) h^s(1)$$

(2) In Theorem 3, if we choose $x = b$, we get

$$\frac{1}{b-a} \int_a^b f(x) dx \leq f(b) h^s(1) + f(a) h^s(0)$$

(3) In Theorem 3, if we choose $x = \frac{a+b}{2}$, we get

$$\frac{1}{b-a} \int_a^b \sqrt{f(x)g(x)} dx \leq \frac{1}{2} h^s\left(\frac{1}{2}\right) (f(a) + f(b) + g(a) + g(b)).$$

Theorem 4. Let $f, g \in SX((h-s)_1, I)$, h is super-multiplicative on I . If $f, g, fg \in L_1[a, b]$ and $h \in L_1[0, 1]$, then we have the following inequality;

$$\begin{aligned}
& (2.4) \\
& \frac{1}{(b-a)^2} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y) g(tx + (1-t)y) dt dx dy \\
& \leq \left(\frac{1}{b-a} \int_a^b f(x) g(x) dx \right) \left(\int_0^1 h^s(t^2) dt \right) + \left(\frac{1}{b-a} \int_a^b f(y) g(y) dy \right) \left(\int_0^1 (1-h^s(t))^2 dt \right) \\
& \quad + \left(\frac{1}{b-a} \int_a^b \int_a^b [f(y)g(x) + f(x)g(y)] dx dy \right) \left(\int_0^1 (h^s(t) - h^s(t^2)) dt \right)
\end{aligned}$$

for all $s \in (0, 1]$, $x, y \in I \subseteq \mathbb{R}$.

Proof. Since f, g are $(h-s)_1$ -convex functions on I , we have

$$\begin{aligned}
f(tx + (1-t)y) & \leq h^s(t)f(x) + (1-h^s(t))f(y) \\
g(tx + (1-t)y) & \leq h^s(t)g(x) + (1-h^s(t))g(y).
\end{aligned}$$

By multiplying the above inequalities and since h is super-multiplicative function, we get

$$\begin{aligned}
& f(tx + (1-t)y) g(tx + (1-t)y) \\
& \leq h^s(t^2)f(x)g(x) + (1-h^s(t))^2 f(y)g(y) \\
& \quad + (h^s(t) - h^s(t^2)) [f(y)g(x) + f(x)g(y)].
\end{aligned}$$

By integrating the resulting inequality with respect to t over $[0, 1]$ and with respect to x, y over $[a, b] \times [a, b]$, we have

$$\begin{aligned}
& \frac{1}{b-a} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y) g(tx + (1-t)y) dt dx dy \\
& \leq \left(\int_a^b f(x) g(x) dx \right) \left(\int_0^1 h^s(t^2) dt \right) + \left(\int_a^b f(y) g(y) dy \right) \left(\int_0^1 (1-h^s(t))^2 dt \right) \\
& \quad + \int_a^b \int_a^b [f(y)g(x) + f(x)g(y)] dx dy \int_0^1 (h^s(t) - h^s(t^2)) dt
\end{aligned}$$

By dividing $\frac{1}{b-a}$, the proof is completed. \square

Remark 1. In Theorem 4, if we take $h(t) = t$ and $s = 1$, then we get

$$\begin{aligned}
& \frac{1}{(b-a)^2} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y) g(tx + (1-t)y) dt dx dy \\
& \leq \frac{1}{3(b-a)} \int_a^b f(x) g(x) dx + \frac{1}{3(b-a)} \int_a^b f(y) g(y) dy + \frac{1}{6(b-a)} \int_a^b \int_a^b N(x, y) dx dy.
\end{aligned}$$

Theorem 5. Let $f, g \in SX((h-s)_2, I)$, h is super-multiplicative and f, g be similarly ordered functions on I . If $f, g, fg \in L_1[a, b]$ and $h \in L_1[0, 1]$, then we have the following inequality;

$$\begin{aligned} & \frac{1}{(b-a)^2} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y) g(tx + (1-t)y) dt dx dy \\ & \leq \left(\frac{1}{b-a} \int_a^b f(x) g(x) dx \right) \left(\int_0^1 (h^s(t^2) + h^s(t-t^2)) dt \right) \\ & \quad + \left(\frac{1}{b-a} \int_a^b f(y) g(y) dy \right) \left(\int_0^1 (h^s((1-t)^2) + h^s(t-t^2)) dt \right) \\ & = \left(\frac{1}{b-a} \int_a^b f(x) g(x) dx \right) \left(\int_0^1 (h^s(t) + h^s(1-t))^2 dt \right) \end{aligned}$$

for all $s \in (0, 1]$, $x, y \in I \subseteq \mathbb{R}$.

Proof. By a similar argument to the proof of the previous Theorem, since f, g are $(h-s)_2$ -convex functions on I , we have

$$\begin{aligned} f(tx + (1-t)y) & \leq h^s(t)f(x) + h^s(1-t)f(y) \\ g(tx + (1-t)y) & \leq h^s(t)g(x) + h^s(1-t)g(y). \end{aligned}$$

By multiplying the above inequalities and since h is super-multiplicative function, we get

$$\begin{aligned} & f(tx + (1-t)y) g(tx + (1-t)y) \\ & \leq h^s(t^2)f(x)g(x) + h^s((1-t)^2)f(y)g(y) \\ & \quad + h^s(t-t^2)[f(y)g(x) + f(x)g(y)]. \end{aligned}$$

Since f and g are similarly ordered functions, we get

$$\begin{aligned} & f(tx + (1-t)y) g(tx + (1-t)y) \\ & \leq (h^s(t^2) + h^s(t-t^2)) f(x)g(x) + (h^s((1-t)^2) + h^s(t-t^2)) f(y)g(y). \end{aligned}$$

By integrating the resulting inequality with respect to t over $[0, 1]$ and with respect to x, y over $[a, b] \times [a, b]$, we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y) g(tx + (1-t)y) dt dx dy \\ & \leq \left(\int_a^b f(x) g(x) dx \right) \left(\int_0^1 (h^s(t^2) + h^s(t-t^2)) dt \right) \\ & \quad + \left(\int_a^b f(y) g(y) dy \right) \left(\int_0^1 (h^s((1-t)^2) + h^s(t-t^2)) dt \right) \\ & = \left(\int_a^b f(x) g(x) dx \right) \left(\int_0^1 (h^s(t) + h^s(1-t))^2 dt \right). \end{aligned}$$

By dividing $\frac{1}{b-a}$, the proof is completed. \square

Theorem 6. Let $f, g \in SX((h-s)_2, I)$, h is super-multiplicative and f, g be similarly ordered functions on I . If $f, g, fg \in L_1[a, b]$ and $h \in L_1[0, 1]$, then we have the following inequality;

$$\begin{aligned} & \frac{1}{(b-a)^2} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y) g(tx + (1-t)y) dt dx dy \\ & \leq \left(\frac{1}{b-a} \int_a^b f(x) g(x) dx \right) \left(\int_0^1 (h^s(t^2) + h^s(t-t^2)) dt \right) \\ & \quad + f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \left(\int_0^1 (h^s((1-t)^2) + h^s(t-t^2)) dt \right) \end{aligned}$$

for all $s \in (0, 1]$, $x, y \in I \subseteq \mathbb{R}$.

Proof. Since f, g are $(h-s)_2$ -convex functions on I , we have

$$\begin{aligned} f\left(tx + (1-t)\frac{a+b}{2}\right) & \leq h^s(t)f(x) + h^s(1-t)f\left(\frac{a+b}{2}\right) \\ g\left(tx + (1-t)\frac{a+b}{2}\right) & \leq h^s(t)g(x) + h^s(1-t)g\left(\frac{a+b}{2}\right). \end{aligned}$$

By multiplying the above inequalities and since h is super-multiplicative function, we get

$$\begin{aligned} & f\left(tx + (1-t)\frac{a+b}{2}\right) g\left(tx + (1-t)\frac{a+b}{2}\right) \\ & \leq h^s(t^2)f(x)g(x) + h^s((1-t)^2)f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ & \quad + h^s(t-t^2) \left[f\left(\frac{a+b}{2}\right)g(x) + f(x)g\left(\frac{a+b}{2}\right) \right]. \end{aligned}$$

Since f and g are similarly ordered functions, we get

$$\begin{aligned} & f(tx + (1-t)y) g(tx + (1-t)y) \\ \leq & (h^s(t^2) + h^s(t-t^2)) f(x)g(x) + \left(h^s((1-t)^2) + h^s(t-t^2) \right) f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right). \end{aligned}$$

By integrating the resulting inequality with respect to t over $[0, 1]$ and with respect to x, y over $[a, b] \times [a, b]$, we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \int_a^b \int_0^1 f(tx + (1-t)y) g(tx + (1-t)y) dt dx dy \\ \leq & \left(\int_a^b f(x)g(x) dx \right) \left(\int_0^1 (h^s(t^2) + h^s(t-t^2)) dt \right) \\ & + (b-a) f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \left(\int_0^1 (h^s((1-t)^2) + h^s(t-t^2)) dt \right). \end{aligned}$$

By dividing $\frac{1}{b-a}$, the proof is completed. \square

Theorem 7. Let $f, g \in SX((h-s)_2, I)$, h is super-multiplicative and f, g be similarly ordered functions on I . If $f, g, fg \in L_1[a, b]$ and $h \in L_1[0, 1]$, then we have the following inequalities;

$$\begin{aligned} (2.5) \quad & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ \leq & 2h^s\left(\frac{1}{4}\right) M(a, b) \int_0^1 (h^s(t) + h^s(1-t))^2 dt. \end{aligned}$$

and

$$\begin{aligned} (2.6) \quad & \frac{1}{2h^{2s}\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ \leq & \frac{1}{b-a} \int_a^b f(x) g(x) dx + \frac{M(a, b)}{2} \left(\int_0^1 (h^s(t) + h^s(1-t))^2 dt \right) \end{aligned}$$

for all $s \in (0, 1]$.

Proof. Since f and g are $(h-s)_2$ -convex on $[a, b]$, then for $t \in [a, b]$, we observe that

$$\begin{aligned} (2.7) \quad & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ = & f\left(\frac{ta+(1-t)b}{2} + \frac{tb+(1-t)a}{2}\right) g\left(\frac{ta+(1-t)b}{2} + \frac{tb+(1-t)a}{2}\right) \\ \leq & h^{2s}\left(\frac{1}{2}\right) [(f(ta+(1-t)b) + f(tb+(1-t)a))] \\ & \times [(g(ta+(1-t)b) + g(tb+(1-t)a))]. \end{aligned}$$

By using $(h-s)_2$ -convexity of f and g , we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq & h^{2s}\left(\frac{1}{2}\right)[(h^s(t)f(a)+h^s(1-t)f(b)+h^s(t)f(b)+h^s(1-t)f(a))] \\ & \times [(h^s(t)g(a)+h^s(1-t)g(b)+h^s(t)g(b)+h^s(1-t)g(a))]. \end{aligned}$$

Since h is super-multiplicative, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq & h^s\left(\frac{1}{4}\right)\left(h^s(t^2)+2h^s(t-t^2)+h^s((1-t)^2)\right) \\ & \times [f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)]. \end{aligned}$$

By using the similarly ordered property of f and g , we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq & 2h^s\left(\frac{1}{4}\right)(h^s(t)+h^s(1-t))^2[f(a)g(a)+f(b)g(b)]. \end{aligned}$$

By integrating both sides respect to t over the interval $[0, 1]$, we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq & 2h^s\left(\frac{1}{4}\right)[f(a)g(a)+f(b)g(b)]\int_0^1(h^s(t)+h^s(1-t))^2dt. \end{aligned}$$

Which completes the proof of the first inequality. Therefore, from the inequality (2.7), we can write

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq & h^{2s}\left(\frac{1}{2}\right)[(f(ta+(1-t)b)+f(tb+(1-t)a))] \\ & \times [(g(ta+(1-t)b)+g(tb+(1-t)a))] \\ \leq & h^{2s}\left(\frac{1}{2}\right)[f(ta+(1-t)b)g(ta+(1-t)b) \\ & +f(tb+(1-t)a)g(tb+(1-t)a)] \\ & +h^{2s}\left(\frac{1}{2}\right)[[h^s(t)f(a)+h^s(1-t)f(b)][h^s(t)g(b)+h^s(1-t)g(a)] \\ & +[h^s(t)f(b)+h^s(1-t)f(a)][h^s(t)g(a)+h^s(1-t)g(b)]]. \end{aligned}$$

Now by using the similar ordered property of f and g , by integrating the resulting inequality, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq & h^{2s}\left(\frac{1}{2}\right)\int_0^1 [f(ta + (1-t)b)g(ta + (1-t)b) \\ & + f(tb + (1-t)a)g(tb + (1-t)a)]dt \\ & + h^{2s}\left(\frac{1}{2}\right)[f(a)g(a) + f(b)g(b)]\left(\int_0^1 (h^s(t) + h^s(1-t))^2 dt\right). \end{aligned}$$

Changing of the variable, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq & \frac{2h^{2s}\left(\frac{1}{2}\right)}{b-a}\int_a^b f(x)g(x)dx \\ & + h^{2s}\left(\frac{1}{2}\right)[f(a)g(a) + f(b)g(b)]\left(\int_0^1 (h^s(t) + h^s(1-t))^2 dt\right). \end{aligned}$$

Dividing both sides of the resulting inequality by $2h^{2s}\left(\frac{1}{2}\right)$, we get the second inequality. \square

Remark 2. In Theorem 7, if we take $h(t) = t$, then we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq & 2^{1-2s}M(a,b)\left(\frac{1}{1+2s} + \frac{\sqrt{\pi}}{2^{2s}}\frac{\Gamma(1+s)}{\Gamma(\frac{3}{2}+s)} + \frac{2^{2s}\sqrt{2}}{2^{2s-1/2}(2+4s)}\right) \\ = & M(a,b)\left(\frac{4}{2^{2s}(1+2s)} + \frac{2\sqrt{\pi}}{2^{4s}}\frac{\Gamma(1+s)}{\Gamma(\frac{3}{2}+s)}\right) \end{aligned}$$

and

$$\begin{aligned} & 2^{2s-1}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq & \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{M(a,b)}{2}\int_0^1 (t^s + (1-t)^s)^2 dt \\ = & \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{M(a,b)}{2}\left(\frac{2}{1+2s} + \frac{\sqrt{\pi}}{2^{2s}}\frac{\Gamma(1+s)}{\Gamma(\frac{3}{2}+s)}\right) \end{aligned}$$

for all $s \in (0, 1]$.

Theorem 8. Let $f, g \in SX((h-s)_2, I)$, h is super-multiplicative and f, g be similarly ordered functions on I . If $f, g, fg \in L_1[a, b]$ and $h \in L_1[0, 1]$, then we have the following inequalities;

$$\begin{aligned} & \frac{g(b)}{b-a} \int_a^b h^s\left(\frac{x-a}{b-a}\right) f(x) dx + \frac{g(a)}{b-a} \int_a^b h^s\left(\frac{b-x}{b-a}\right) f(x) dx \\ & + \frac{f(b)}{b-a} \int_a^b h^s\left(\frac{x-a}{b-a}\right) g(x) dx + \frac{f(a)}{b-a} \int_a^b h^s\left(\frac{b-x}{b-a}\right) g(x) dx \\ & \leq \frac{1}{b-a} \int_a^b f(x) g(x) dx + f(b) g(b) \int_0^1 [h^s(t-t^2) + h^s(t^2)] dt \\ & + f(a) g(a) \int_0^1 [h^s((1-t)^2) + h^s(t-t^2)] dt. \end{aligned}$$

for all $s \in (0, 1]$.

Proof. Since f and g are $(h-s)_2$ -convex functions, we can write

$$f(tb + (1-t)a) \leq h^s(t)f(b) + h^s(1-t)f(a)$$

and

$$g(tb + (1-t)a) \leq h^s(t)g(b) + h^s(1-t)g(a)$$

By using the elementary inequality, $e \leq f$ and $p \leq r$, then $er + fp \leq ep + fr$ for $e, f, p, r \in \mathbb{R}$, then we get

$$\begin{aligned} & f(tb + (1-t)a) [h^s(t)g(b) + h^s(1-t)g(a)] \\ & + g(tb + (1-t)a) [h^s(t)f(b) + h^s(1-t)f(a)] \\ & \leq f(tb + (1-t)a) g(tb + (1-t)a) \\ & + [h^s(t)f(b) + h^s(1-t)f(a)] [h^s(t)g(b) + h^s(1-t)g(a)]. \end{aligned}$$

So, we obtain

$$\begin{aligned} & h^s(t)f(tb + (1-t)a)g(b) + h^s(1-t)f(tb + (1-t)a)g(a) \\ & + h^s(t)f(b)g(tb + (1-t)a) + h^s(1-t)f(a)g(tb + (1-t)a) \\ & \leq f(tb + (1-t)a)g(tb + (1-t)a) \\ & + [h^{2s}(t)f(b)g(b) + h^s(t)h^s(1-t)f(b)g(a) + h^s(t)h^s(1-t)f(a)g(b) + h^{2s}(1-t)f(a)g(a)]. \end{aligned}$$

By using similarly ordered property of f and g , we have

$$\begin{aligned} & h^s(t)f(tb + (1-t)a)g(b) + h^s(1-t)f(tb + (1-t)a)g(a) \\ & + h^s(t)f(b)g(tb + (1-t)a) + h^s(1-t)f(a)g(tb + (1-t)a) \\ & \leq f(tb + (1-t)a)g(tb + (1-t)a) \\ & + [h^{2s}(1-t) + h^s(t)h^s(1-t)]f(a)g(a) + [h^s(t)h^s(1-t) + h^{2s}(t)]f(b)g(b). \end{aligned}$$

Since h is super-multiplicative, we can write

$$\begin{aligned} & h^s(t)f(tb + (1-t)a)g(b) + h^s(1-t)f(tb + (1-t)a)g(a) \\ & + h^s(t)f(b)g(tb + (1-t)a) + h^s(1-t)f(a)g(tb + (1-t)a) \\ \leq & f(tb + (1-t)a)g(tb + (1-t)a) \\ & + [h^s((1-t)^2) + h^s(t-t^2)]f(a)g(a) + [h^s(t-t^2) + h^s(t^2)]f(b)g(b). \end{aligned}$$

By integrating this inequality with respect to t over $[0, 1]$ and by using the change of the variable $tb + (1-t)a = x$, $(b-a)dt = dx$, the proof is completed. \square

3. APPLICATIONS TO SOME SPECIAL MEANS

We shall consider the means as arbitrary positive real numbers a, b , $a \neq b$.

The geometric mean:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0,$$

The Identric mean.

$$I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}, \quad a, b \geq 0,$$

Now, we present some applications of the result in section 2 to the special means of real numbers. The following propositions hold:

Proposition 1. Let $2 < a < b < \infty$. Then for all $s \in (0, 1]$, we have

$$I(a, b) \leq G^{\frac{2}{s+1}}(a, b).$$

Proof. The proof is immediate follows from Theorem 2 applied for $f(x) = g(x) = \ln x$; $x \in [2; \infty)$ and $h(t) = t$. In other words, the following is obtained.

$$\begin{aligned} & \frac{1}{b-a} \int_a^b \ln x dx = \ln I(a, b) \\ \leq & \frac{\ln b}{(b-a)^{2+s}} \int_a^b (x-a)(x-a)^s dx + \frac{\ln a}{(b-a)^{2+s}} \int_a^b (x-a)(b-x)^s dx \\ & + \frac{\ln b}{(b-a)^{2+s}} \int_a^b (b-x)(x-a)^s dx + \frac{\ln a}{(b-a)^{2+s}} \int_a^b (b-x)(b-x)^s dx \\ = & \frac{\ln a + \ln b}{s+2} + \frac{\ln a + \ln b}{(s+1)(s+2)} = \frac{\ln a + \ln b}{s+1} = \ln G^{\frac{2}{s+1}}(a, b) \end{aligned}$$

This completes the proof. \square

Is similar to the above features, may be different applications. We stay away from getting into these details here. Interested readers can add new ones to these applications. The theory of convex functions is expanding day by day. The reason, researchers are opening a new chapter in the theory of convex functions with each passing day. Convex function is obtained as a result of studies of different classes often resemble each other in terms of several properties. Covers all of the previous definitions of new classes sometimes obtained, and sometimes carries a few features.

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